Modules with chain conditions up to isomorphism and artinian dimension

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Joint work with Zahra Nazemian

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F. and Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra **453** (2016), 578–601.

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We study modules with chain conditions up to isomorphism, in the following sense.

Classical definition

Module = a right module M_R over a fixed associative ring R with identity $1 \neq 0$.

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A right module M is *artinian* if, for every descending chain $M \geq M_1 \geq M_2 \geq \dots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n = M_i$ for every $i \geq n$.

New "iso" definition

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A right module M is *isoartinian* if, for every descending chain $M \geq M_1 \geq M_2 \geq \dots$ of submodules of M, there exists an index $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq n$.

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Examples: $\mathbb{Z}_{\mathbb{Z}}$, *any vector space* V_k *over a field* k.

Right artinian rings

Classical definition: A ring R is *right artinian* if R_R is an artinian module.

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Iso definition: A ring R is *right isoartinian* if R_R is an isoartinian module.

(Iso)noetherian modules

Classical definition: A module M is *noetherian* if, for every ascending chain $M_1 \leq M_2 \leq \dots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n = M_i$ for every $i \geq n$.

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Iso definition: A module M is *isonoetherian* if, for every ascending chain $M_1 \leq M_2 \leq \ldots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$.

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We find in these "iso" notions a surprising analogy with the classical case of artinian, noetherian and simple modules.

Standard characterizations of noetherian modules

Classical result:

Proposition

The following conditions are equivalent for a right module M:
(i) *M is noetherian (that is, for every ascending chain $M_1 \leq M_2 \leq \dots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n = M_i$ for every $i \geq n$.)*
(ii) *Every non-empty chain of submodules of M has a greatest element.*
(iii) *Every nonempty set of submodules of M has a maximal element.*

Characterizations of isonoetherian modules

"Iso" result:

Proposition

The following conditions are equivalent for a right module M:
(i) *M is isonoetherian (that is, for every ascending chain $M_1 \leq M_2 \leq \dots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$.)*

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(ii) *For every non-empty chain $\mathcal{C}$ of submodules of M, there exists $N \in \mathcal{C}$ such that, for every $N' \geq N$, if $N' \in \mathcal{C}$, then $N \cong N'$.*

Characterizations of isonoetherian modules

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(ii) *For every non-empty chain $\mathcal{C}$ of submodules of M, there exists $N \in \mathcal{C}$ such that, for every $N' \geq N$, if $N' \in \mathcal{C}$, then $N \cong N'$.*
(iii) *For every non-empty set $\mathcal{F}$ of submodules of M, there exists $N \in \mathcal{F}$ such that, for every $N' \geq N$, if $N' \in \mathcal{F}$, then $N \cong N'$.*

Artinian modules are essential extensions of their socle

Classical result:

Theorem

Any artinian module M contains an essential submodule that is a direct sum of simple modules.

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Iso result:

Theorem

Any isoartinian module M contains an essential submodule that is a direct sum of isosimple modules.

Non-zero endomorphisms of simple modules

Classical result:

Lemma

Every non-zero endomorphism of a simple module is a bijective mapping.

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Lemma
Every non-zero endomorphism of an isosimple module is an injective mapping.

Endomorphism rings of simple modules

Classical result (Schur's Lemma):

Lemma

The endomorphism ring of a simple module is a division ring.

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Iso result:

Lemma

The endomorphism ring of an isosimple right module is a right Ore domain, whose principal right ideals form a noetherian modular lattice with respect to inclusion.

Semiprime ring

A ring R is *semiprime* if for every two-sided ideal I of R and every integer $n > 0$, $I^n = 0$ implies $I = 0$.

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A commutative ring R is semiprime if and only if it has no non-zero nilpotent elements $(= R$ is *reduced*$)$.

The Artin-Wedderburn Theorem (semiprime right artinian rings)

Classical result:

Theorem

The following conditions are equivalent for a ring R.

(i) *R is a semiprime right artinian ring.*

(ii) *Every right R-module is projective.*

(iii) *Every right R-module is injective.*

(iv) *Every right R-module is semisimple.*

The Artin-Wedderburn Theorem (semiprime right artinian rings)

(v) Every short exact sequence of right R-modules splits.

(vi) The module R_R is semisimple, i.e., a (direct) sum of simple right ideals.

(vii) R is right artinian and has no non-zero nilpotent right ideal.

(viii) There exist integers $t, n_1, \ldots, n_t \geq 1$ and division rings $D_1, \ldots, D_t$ such that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t). \quad (1)$$

The corresponding iso result for semiprime right isoartinian rings

Theorem

The following conditions are equivalent for a semiprime right isoartinian ring R:

(i) *R is right noetherian.*

(ii) *R is right Goldie.*

(iii) *R_R has finite Goldie dimension.*

(iv) *If S is an isosimple right ideal of R and I_S the ideal which is the sum of all the right ideals isomorphic to S, then I_S is a finite direct sum of isosimple right ideals.*

(v) *R_R is a (direct) sum of isosimple right ideals.*

The Wedderburn Theorem

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(The Wedderburn Theorem) *A ring R is simple and right artinian if and only if R is a matrix ring over a division ring.*

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Iso result:

Theorem

A ring R is simple and right isoartinian if and only if R is a matrix ring over a simple principal right ideal domain.

Commutative semiprime rings

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Proposition

A commutative semiprime ring is isoartinian if and only if it is isomorphic to a finite direct product of principal ideal domains.

Noetherian modules and Goldie dimension

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Every noetherian module has finite Goldie dimension.

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Noetherian modules and Goldie dimension

Classical result (Goldie):

Theorem

An integral domain D is right Ore if and only if D_D is uniform, if and only if D_D has finite Goldie dimension. In particular, right noetherian domains are right Ore domains.

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An integral domain D is right Ore if and only if D_D is uniform, if and only if D_D has finite Goldie dimension. In particular, right noetherian domains are right Ore domains.

Iso result:

Theorem

If D is a right isonoetherian domain, then D is a right Ore domain.

Krull dimension

Classical definition:
The class $\mathcal{K}_\alpha$ of modules *of Krull dimension* α is defined as follows.

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(a) $M \notin \bigcup_{\beta<\alpha} \mathcal{K}_\beta$;

(b) for every countable descending chain $A_0 \geq A_1 \geq A_2 \geq \dots$ of submodules of M, there exists an index n such that the factors A_i/A_{i+1} belong to $\bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for every $i \geq n$.

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If $M \notin \mathcal{K}_\alpha$ for every α one says that M *fails to have Krull dimension*, or that *K.dim*(M_R) *is not defined.*

Thus, $\text{K.dim}(M_R) = -1$ if and only if $M_R = 0$

$\text{K.dim}(M_R) = 0$ if and only if M_R is a non-zero artinian module.

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Let R be a ring and $\mathcal{A}$ be the class of all artinian right R-modules. Let $\mathcal{A}_0$ be the set containing only the zero module, and set $\mathcal{A}_1 := \mathcal{A}$. Define by induction, for every ordinal number $\alpha > 1$, $\mathcal{A}_\alpha$ to be the class of all the right R-modules M for which, for every submodule N of M, either $N \cong M$ or $N \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. If a module M belongs to some $\mathcal{A}_\alpha$, we say that M *has artinian dimension* and the least such α is the *artinian dimension* of M, denoted by $\operatorname{art.dim}(M)$. If M does not belong to any $\mathcal{A}_\alpha$, we say that M *does not have artinian dimension.*

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Artininan dimension

Thus:

(1) The zero module is of artinian dimension 0.

(2) Artinian modules are of artinian dimension 1.

(3) If a module M has artinian dimension, then its submodules have artinian dimension $\leq \operatorname{art.dim}(M)$.

Artininan dimension, isoartinian modules, semisimple artinian rings

Theorem

A module has artinian dimension if and only if it is isoartinian.

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Theorem

A ring R is semisimple artinian if and only if all right R-modules have artinian dimension.

(Jacobson) radical and isoradical of a ring

Classical definition:
The *(right Jacobson) radical* $\mathrm{rad}(R_R)$ of a ring R is the intersection of the annihilators of all simple right R-modules.

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The *right isoradical* $\text{I-rad}(R_R)$ of a ring R is the intersection of the annihilators of all isosimple right R-modules.

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Clearly:
(1) $\text{I-rad}(R_R)$ is a two-sided ideal of R.

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Clearly:
(1) $\text{I-rad}(R_R)$ is a two-sided ideal of R.
(2) $\text{I-rad}(R_R) \subseteq \text{rad}(R)$ of R.

But. . .

There exist right noetherian right chain domains in which the right isoradical and the left isoradical are different.

Radical zero

Classical result:

Proposition

A right artinian ring R is semiprime if and only if its (right Jacobson) radical $\text{rad}(R)$ *is zero.*

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Socle and isosocle

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Let $\mathcal{U}$ be any class of right R-modules and $\text{Gen}(\mathcal{U})$ the class of all right modules M_R for which there exist an indexed set $(U_\alpha)_{\alpha \in A}$ in $\mathcal{U}$ and an epimorphism $\oplus_{\alpha \in A} U_\alpha \to M$.

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Set $\text{Tr}_M(\mathcal{U}) := \sum \{ h(U) \mid h \colon U \to M \text{ is a homomorphism for some } U \in \mathcal{U} \}$. Thus $M \in \text{Gen}(\mathcal{U})$ if and only if $\text{Tr}_M(\mathcal{U}) = M$.

Socle and isosocle

Classical definition: For $\mathcal{U}$ the class of all simple right R-modules, $\mathrm{Tr}_M(\mathcal{U}) = \mathrm{soc}(M)$, the socle of M_R.

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Iso definition: For $\mathcal{U}$ the class of all isosimple right R-modules, $\mathrm{Tr}_M(\mathcal{U}) := \text{I-soc}(M)$, the I-*socle* of M_R.

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(1) $\mathrm{soc}(M_R) \leq \text{I-soc}(M)$.

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Iso definition: For $\mathcal{U}$ the class of all isosimple right R-modules, $\mathrm{Tr}_M(\mathcal{U}) := \text{I-soc}(M)$, the I-*socle* of M_R.

(1) $\mathrm{soc}(M_R) \leq \text{I-soc}(M)$.

(2) If M_R is a nonsingular module, then $\text{I-soc}(M_R) = M_R$ if and only if M_R is the sum of its isosimple submodules.

Semisimple artinian rings again

Classical result:

Theorem

The following conditions are equivalent for a ring R:

(1) *R is semisimple artinian.*

(2) $\mathrm{soc}(R) = R$.

(3) *R is a sum of simple right ideals.*

(4) *For any right R-module M, $\mathrm{soc}(M) = M$.*

(5) *R is a finite direct product of matrix rings over division rings.*

(6) *R is a direct sum of simple right ideals.*

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Iso result:

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The following conditions are equivalent for a ring R:
(1) $\text{I-soc}(R) = R$.
(2) *R is a sum of isosimple right ideals.*
(3) *For any right R-module M, $\text{I-soc}(M) = M$.*
(4) *R is a finite direct product of prime right noetherian rings, each of which is a sum of isosimple right ideals.*

Semisimple artinian rings again

Theorem

The following conditions are equivalent for a ring R:

(1) *R is a finite direct product of matrix rings over principal right ideal domains.*

(2) *R is a direct sum of isosimple right ideals.*

(3) *R is right semihereditary and* $\text{I-soc}(R_R) = R$.

(4) *R is a semiprime right isoartinian right noetherian ring.*

(5) *R is a semiprime right isoartinian ring and R_R is of finite uniform dimension.*

Modules of finite I-length

We say that a chain $0 = P_0 < P_1 < \cdots < P_n = M$ of submodules of M is an I-*series* for M if $P_i \ncong P_{i+1}$ for each i and, for every submodule K of M with $P_i \leq K \leq P_{i+1}$, we have that either $K \cong P_i$ or $K \cong P_{i+1}$.

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Modules of finite I-length

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Modules of finite length, modules of finite l-length

Classical result:

Proposition

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Iso result:

Modules of finite length, modules of finite I-length

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We don't know if the converse of this proposition holds. We only have some partial results.

Some things are different

Proposition

If R is right artinian (right noetherian), then every finitely generated right R-module is right artinian (right noetherian).

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If R is right artinian (right noetherian), then every finitely generated right R-module is right artinian (right noetherian).

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If R is right artinian (right noetherian), then every finitely generated right R-module is right artinian (right noetherian).

But:

There exist right isoartinian rings with cyclic right modules that are not isoartinian.